Computing the modified Legendre functions (also relevant for computing spherical harmonic functions).

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Let $P_l^m(x)$ be the Legendre function. Press (1992, Numerical Recipes, 2nd edition, page 246-248) showed that $P_l^m(x)$ can be computed using a recursion relation (eqn 6.8.7 in Press [1992]):

$$(l-m)P_l^m = x(2l-1)P_{l-1}^m - (l+m-1)P_{l-2}^m, \qquad (1)$$

coupled with two other equations (eqns 6.8.8 and 6.8.9):

$$P_m^m = (-1)^m(2m-1)!!(1-x^2)^{m/2}, \qquad (2)$$

$$P_{m+1}^m = x(2m+1)P_m^m. \qquad (3)$$

Often in defining spherical harmonic functions (e.g., basis functions used for spherical harmonic expansion, a factor involving factorials is multiplied to $P_l^m(x)$. For example, in CitcomS, we use a modified Legendre function $\bar{p}_l^m(x)$ as basis functions for spherical harmonic expansion [Zhong et al., 2008]. $\bar{p}_l^m(x)$ is defined as

$$\bar{p}_l^m(x) = \sqrt{\frac{(2-\delta_{0m})(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(x). \qquad (4)$$

Directly multiplying $P_l^m(x)$ with the factor in the above expression leads to difficulties for large *l*. One way to fix the problem is to derive another recursion relation from eqn 1 using $\bar{p}_l^m(x)$.

For simplicity, let us ignore $\sqrt{\frac{(2-\delta_{0m})}{4\pi}}$ in eqn 4 first. That is, we only consider

$$\bar{p}_l^m(x) = \sqrt{\frac{(2l+1)(l-m)!}{(l+m)!}}P_l^m(x). \qquad (5)$$

Multiplying eqn 1 with $\sqrt{\frac{(2l+1)(l-m)!}{(l+m)!}}$ leads to

$$(l-m)\bar{p}_l^m = x\sqrt{\frac{(4l^2-1)(l-m)}{l+m}}\bar{p}_{l-1}^m - \sqrt{\frac{(2l-1)(l-m)(l-1-m)(l-1+m)}{(2l-3)(l+m)}}\bar{p}_{l-2}^m\,. \qquad (6)$$

Eqn (6) is the recursion relation used in CitcomS in computing $\bar{p}_l^m(x)$.
Eqns (2) and (3) are also needed to modified in connection to eqn (6). They are:

$$\bar{p}_m^m = (-1)^m\sqrt{\frac{(2m+1)!!}{(2m)!!}}(1-x^2)^{m/2}, \qquad (7)$$

$$\bar{p}_{m+1}^m = x\sqrt{2m+3}\,\bar{p}_m^m. \qquad (8)$$

Eqns (6)-(8) are used in function: double modified_glgndr_a (*l*,*m*,*t*).